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Frequency Sampling Filters— Hilbert Transformers and Resonators

By R. E. BOGNER

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We first briefly review the principles of frequency sampling filters. We also show that the "conventional" frequency sampling filter can be modified simply to give an output which is the Hilbert transform of the original output. Both the original and transformed outputs are made available by the use of the simple complex number resonator described. The relationship between this system and filtering by Fourier transforming is shown.

I. INTRODUCTION

Frequency sampling filters are filters whose frequency responses are synthesized as the sum of elemental frequency responses of the form (Fig. 1a)[1]

$$V_k(f) = A_k \frac{\sin [\pi(f - f_k)/f_o]}{\pi(f - f_k)/f_o} e^{-i2\pi f\tau} + A_k \frac{\sin \pi(f + f_k)/f_o}{\pi(f + f_k)/f_o} e^{-i2\pi f\tau} \quad (1)$$

where

$V_k(f)$ is the transfer function of the kth response;
A_k is a constant multiplier, the value of the amplitude response at frequency f_k;
f is frequency in hertz;
f_k is the kth sampling frequency $= kf_o$;

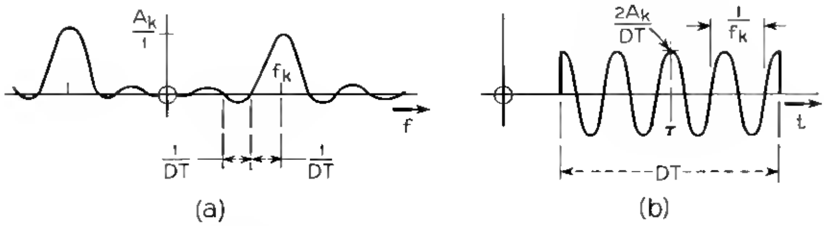

Fig. 1 — (a) Elemental frequency response contribution; (b) Elemental time response contribution.

f_o is the frequency interval between samples, that is, $f_o = f_{k+1} - f_k$, $f_o = 1/DT$, D = delay in samples;

τ is the group delay, a constant for all the responses.

Because of the constant group delay, the amplitude versus frequency response, $|V(f)|$, of the sum is given by

$$| V(f) | = \sum_k A_k \left[\frac{\sin \pi(f - f_k)/f_o}{\pi(f - f_k)/f_o} + \frac{\sin \pi(f + f_k)/f_o}{\pi(f + f_k)/f_o} \right] \cdots . \tag{2}$$

By choice of the A_k, suitable amplitude responses for many applications may be specified. These will be bandlimited functions of frequency.

The elemental time responses, $v_k(t)$ (Fig. 1b) are convenient to realize by digital methods. They are truncated cosine waves.

Figure 2 shows a comb filter, whose impulses occur DT seconds apart, followed by a resonator, whose impulse response is a cosine wave of frequency an integral multiple of $1/DT$. The overall impulse response is the sum of the cosine responses to the two impulses; this is zero before the positive impulse, a cosine from then until DT seconds later, and thereafter zero, when the two cosines cancel.

A complete frequency sampling filter is shown in the left of Fig. 3. Usually the resonators have been programmed as conventional second

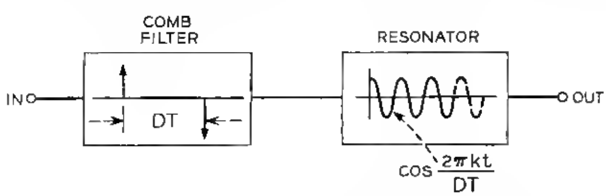

Fig. 2 — Comb filter followed by cosine resonator.

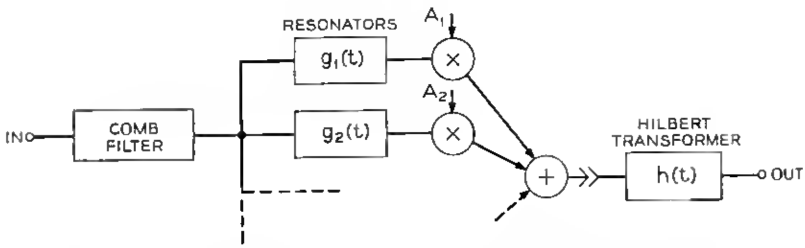

Fig. 3 — Frequency sampling filter, followed by Hilbert transformer.

order systems, with slight damping to ensure stability under conditions of error in the resonator coefficients.

II. USE AS HILBERT TRANSFORMER

A frequency sampling filter may be readily adapted to give an output which is the Hilbert transform of that of the filter described above. Consider the sampling filter (Fig. 3) followed by a Hilbert transformer, $h(t)$. This is equivalent to the system of Fig. 4, where the one Hilbert transformer has been replaced by one at the output of each elemental filter. Now, in the original frequency sampling filter, the kth resonator has an impulse response, for time sampled systems

$$g_k(nT) = \cos \omega_k(nT), \; n = 0, 1, 2, \cdots$$

where T is the sampling interval. The Hilbert transformed version of this is approximately

$$\hat{g}_k(nT) = \sin \omega_k(nT).$$

The approximation is discussed in Appendix A. Thus to make a system equivalent to the original frequency sampling filter plus Hilbert transformer, we need only replace the resonators by ones with impulse responses $\sin \omega_k t$. This could be done by use of modified second order delay resonators; but the system of Fig. 5 is more convenient programwise and is helpful conceptually. This system has the z transform system function

$$\frac{W(z)}{U(z)} = G(z) = \frac{1}{1 - z^{-1} \exp[(\alpha + j\omega)T]} \tag{3}$$

and corresponding impulse response

$$g(nT) = e^{\alpha nT} e^{j\omega nT}, \; n = 0, 1, \cdots . \tag{4}$$

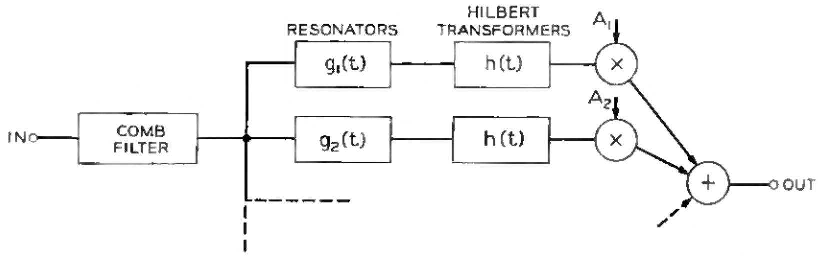

Fig. 4 — Frequency sampling filter with separate Hilbert transformers.

For $\alpha = 0$, the real and imaginary parts are $\cos \omega nT$ and $\sin \omega nT$. A small negative value of α would be used for stability.

The frequency sampling filter then has the form of Fig. 4, with each channel containing one complex number resonator instead of the resonator plus Hilbert transformer. The output at each sampling time is a complex number, whose real part corresponds to the output of a conventional frequency sampling filter, and whose imaginary part is an approximation to the Hilbert transform of the real part.

In Appendix A, the analysis of the approximation results in the following observations:

(*i*) The Hilbert transformer cannot handle signals with frequencies tending to zero.

(*ii*) For signals with low-frequency components, care is necessary in specifying the frequency samples to ensure that the negative-frequency tail of the positive-frequency response component is of small amplitude.

(*iii*) The errors are in the amplitude and not phase characteristics.

The system is capable of filtering a complex input, $u + jv$ without modification of the resonators.

III. RELATION TO DISCRETE FOURIER TRANSFORM

Consider $\alpha = 0$. The response of the kth resonator at time nT, $n = 0, 1, 2, \cdots$, to a unit pulse at time mT is $\exp[j\omega_k(n - m)T]$. Hence the response at time nT to a signal $s(mT)$, $m = \cdots, -1, 0, 1, 2, \cdots$ is:

$$\begin{aligned} x_k(nT) + jy_k(nT) &= \sum_{m=-\infty}^{n} s(mT) \exp[j\omega_k(n - m)T] \\ &= \exp(j\omega_k nT) \sum_{m=-\infty}^{n} s(mT) \exp(-j\omega_k mT). \end{aligned} \tag{5}$$

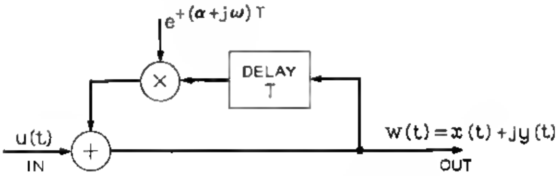

Fig. 5 — Complex number resonator.

When the comb filter precedes the resonator, the effect of its negative impulse, occurring DT seconds after the positive impulse is to add the second term of (6):

$$\begin{aligned} x_k(nT) + jy_k(nT) &= \exp(j\omega_k nT) \sum_{m=-\infty}^{n} s(mT) \exp(-j\omega_k mT) \\ &\qquad - \exp(j\omega_k nT) \sum_{m=-\infty}^{n} s(m - D)T \exp(-j\omega_k mT) \\ &= \exp(j\omega_k nT)\Big[\sum_{m=-\infty}^{n} s(mT) \exp(-j\omega_k mT) \\ &\qquad - \sum_{m=-\infty}^{n-D} s(mT) \exp(-j\omega_k mT) \exp(-j\omega_k DT) \Big]. \end{aligned} \tag{6}$$

But DT is an integral multiple of the period $2\pi/\omega_k$ as mentioned in Section I; thus $\exp(-j\omega_k DT) = 1$. Hence

$$x_k(nT) + jy_k(nT) = \exp(j\omega_k nT) \sum_{m=n-D+1}^{n} s(mT) \exp(-j\omega_k mT). \tag{7}$$

This expression may be recognized as an oscillation $\exp(j\omega_k nT)$ whose coefficient is the value at frequency ω_k of the Discrete Fourier Transform (DFT) of $s(mT)$, computed over the last D samples. The output of the frequency sampling filter, taking into account the weights A_k, is

$$\begin{aligned} x(nT) + jy(nT) &= \sum_k A_k[x_k(nT) + jy_k(nT)] \\ &\sum_k \exp(j\omega_k nT) A_k \sum_{m=n-D+1}^{n} s(mT) \exp(-j\omega_k mT). \end{aligned} \tag{8}$$

This is the Fourier synthesis (inverse DFT) of the frequency function

$$A_k \sum_{m=n-D+1}^{n} s(mT) \exp(-j\omega_k mT), \qquad k = 1, 2, \cdots, \tag{9}$$

which may be regarded as the product of the running DFT of $s(mT)$ and a DFT whose values at frequencies ω_k are the A_k.

Frequency sampling filtering is thus equivalent to filtering by Fourier transforming, multiplying by a filter frequency function, and inverse transforming.

The filter frequency function (A_k, $k = 1, 2, \ldots$) has, so far, been considered real. There is no reason why the A_k should not be complex, permitting the filter to have an arbitrary phase characteristic. The complex values of the A_k may be specified in cartesian or polar form, the latter being more convenient for amplitude-phase specification.

Another way of looking at the resonator output is obtained by rearranging (7):

$$x_k(nT) + jy_k(nT) = \sum_{m=-(D-1)}^{0} s[(m+n)T] \exp(-j\omega_k mT). \qquad (10)$$

This may be recognized as the DFT of the last D values of $s(mT)$, shifted in time so that the latest occurs at time $mT = 0$.

IV. CONCLUSION

The use of complex number resonators in a frequency sampling filter provides a Hilbert transformed output as well as the conventional filtered output. The system can readily accept a complex time function as input, and has a very simple flow chart. The output is equivalent to that obtained by the use of Fourier transforms to perform filtering in the frequency domain.

A sampling filter subroutine using the ideas presented has been written in Fortran IV. It has been used for filtering and Hilbert transforming speech signals in a number of tasks.

V. ACKNOWLEDGMENT

Thanks are due to C. H. Coker, L. R. Rabiner and R. W. Schafer for many helpful discussions. A recursive generation of the DFT is given in Ref. 2.

APPENDIX A

Errors in the Hilbert Transformer

A cosine wave, truncated in time, is the basis of the frequency sampling filters. A correspondingly truncated sine wave has been used as an approximation to the Hilbert transform of the cosine. The errors in this approximation will be analyzed by comparing the

Fourier transform of the truncated sine wave with that of the true Hilbert transform of the cosine. The analysis is for continuous (that is, nonsampled) sines and cosines.

The truncated cosine response is taken to be

$$\begin{aligned} h_c(t) &= \cos \frac{2\pi N t}{T}, & -\frac{T}{2} \leqq t \leqq \frac{T}{2} \\ &= 0, & \text{elsewhere.} \end{aligned}$$

The F transform of $h_c(t)$ is

$$H_c(f) = \frac{T}{2}\left[\frac{\sin \pi T\left(f - \frac{N}{T}\right)}{\pi T\left(f - \frac{N}{T}\right)} + \frac{\sin \pi T\left(F + \frac{N}{T}\right)}{\pi T\left(f + \frac{N}{T}\right)}\right] \tag{11}$$

$$= H_{c1}(f) + H_{c2}(f), \quad \text{respectively.} \tag{12}$$

$H_c(f)$ may be separated further into main responses and "tails" (Fig. 6):

$$H_c(f) = H_{c1+}(f) + H_{c1-}(f) + H_{c2+}(f) + H_{c2}(f) \tag{13}$$

where

$$H_{c1+} = H_{c1}\,, \quad f > 0; \quad \frac{H_{c1}(0)}{2}\,, \quad f = 0; \quad 0, \quad f < 0$$

$$H_{c1-} = 0, \quad f > 0; \quad \frac{H_{c1}(0)}{2}\,, \quad f = 0; \quad H_{c1}\,, \quad f < 0$$

$$H_{c2+} = H_{c2}\,, \quad f > 0; \quad \frac{H_{c2}(0)}{2}\,, \quad f = 0; \quad 0, \quad f < 0$$

$$H_{c2-} = 0, \quad f > 0; \quad \frac{H_{c2}(0)}{2}\,, \quad f = 0; \quad H_{c2}\,, \quad f < 0.$$

The F transform of the Hilbert transform $[\hat{h}_c(t)]$ of $h_c(t)$ is then

$$\hat{H}_c(f) = -j \operatorname{sgn}(f) H_c(f) \tag{14}$$

$$= -jH_{c1+}(f) + jH_{c1-}(f) - jH_{c2+}(f) + jH_{c2-}(f). \tag{15}$$

The truncated sine response is taken to be

$$\begin{aligned} h_s(t) &= \sin \frac{2\pi N t}{T}, & -\frac{T}{2} \leqq t \leqq \frac{T}{2} \\ &= 0, & \text{elsewhere.} \end{aligned}$$

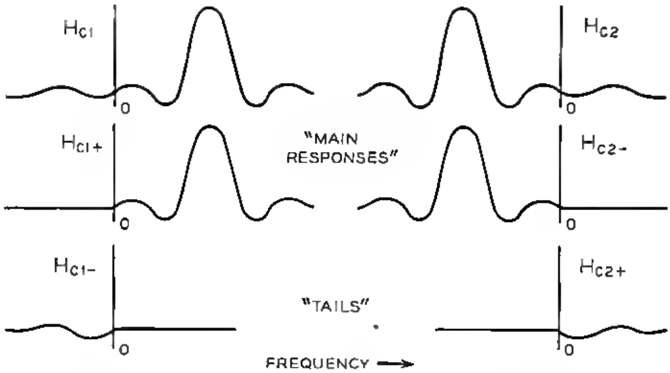

Fig. 6 — Components of elemental frequency response.

The F transform of $h_s(t)$ is

$$H_s(f) = \frac{T}{2}\left[-j\frac{\sin \pi T\left(f-\frac{N}{T}\right)}{\pi T\left(f-\frac{N}{T}\right)} + j\frac{\sin \pi T\left(f+\frac{N}{T}\right)}{\pi T\left(f+\frac{N}{T}\right)}\right] \tag{16}$$

which by comparison with (11), (12), (13) is seen to be

$$\begin{aligned} H_s(f) &= -jH_{c1}(f) + jH_{c2}(f) \\ &= -jH_{c1+}(f) - jH_{c1-}(f) + jH_{c2+}(f) + jH_{c2-}(f). \end{aligned} \tag{17}$$

Then from (15) and (17):

$$H_s(f) = \hat{H}_c(f) - 2jH_{c1-}(f) + 2jH_{c2+}(f). \tag{18}$$

The error in approximating $\hat{H}_c(f)$ by $H_s(f)$ is thus attributable to the tails $H_{c1-}(f)$ and $H_{c2+}(f)$, which are small for $N \gg 1$. From the definitions (11), (12), (13), it follows that these tails are related:

$$H_{c1-}(-f) = H_{c2+}(f). \tag{19}$$

In a complete frequency sampling filter, the transforms corresponding to all the time responses are to be added. Errors in the "Hilbert transformed" output, y, as compared with the straight filtered output, x, are determined by the resultant tails; these tails may be of small amplitude if suitable values are chosen for the frequency samples.

Just what criterion of smallness should be applied depends on the application. Some general observations may be made, however:

(*i*) The Hilbert transformer cannot be useful to zero frequency because a zero frequency sample has tails equal to the main responses, and would thus contribute gross errors. This is of course consistent with the infinite duration of the impulse response $(1/t)$ of a true Hilbert transformer.

(*ii*) To transform signals with low frequency components, many frequency samples may be required to provide the sharp and continued cutoff required for tail suppression.

(*iii*) Since $H_{c1-}(-f) = H_{c2+}(f)$, it follows from (18) that the errors, associated with $H_{c1-}(-f)$ and $H_{c2+}(f)$ are directly in or out of phase with the relevant main responses. The error in the Hilbert transform is thus an amplitude and not a phase error. This result is also consistent with the observation that the approximate Hilbert transformed response to an impulse is truly odd.

APPENDIX B

Relationship between Complex Number Resonator and Conventional Second Order Resonator

While the formal transform relation between (3) and (4) is readily shown, it is satisfying to explain how the seemingly first order delay system can produce an oscillatory response. The system of Fig. 5 is described by the equation

$$x(mT) + jy(mT) = u(mT) + e^{(\alpha+j\omega)T}[x(m-1)T + jy(m-1)T] \quad (20)$$

When a pulse $u(0) = 1$, with zero before and after is applied, the first response is

$$x(0) + jy(0) = 1 + j0$$

The next response is simply the first response multiplied by $e^{(\alpha+j\omega)T}$

$$x(1T) + jy(1T) = e^{(\alpha+j\omega)T}(1 + j0);$$

there is a similar multiplication at each subsequent sampling instant, yielding the impulse response

$$x(nT) + jy(nT) = e^{n(\alpha+j\omega T)}, n = 0, 1, 2 \cdots, \quad (21)$$

equivalent to (4).

The complex number resonator may be shown to contain a second order delay feedback, making its oscillatory response consistent with that of the more conventional second-order systems. Its equation (20)

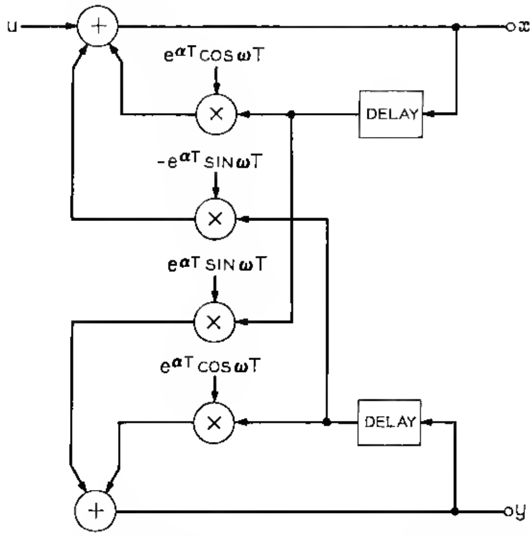

Fig. 7 — Expanded flow chart for complex number resonator.

may be examined by equating separately real and imaginary parts:

$$x(mT) = u(mT) + (e^{\alpha T}\cos\omega T)x[(m-1)T] - (e^{\alpha T}\sin\omega T)y[(m-1)T] \qquad (22)$$

$$y(mT) = (e^{\alpha T}\sin\omega T)x[(m-1)T] + (e^{\alpha T}\cos\omega T)y[(m-1)T] \qquad (23)$$

Equations (22) and (23) may be represented by the flow chart of Fig. 7. There is, in fact, a path of delay two sampling intervals from the real output x, via y, the imaginary part of the output, back to x. Thus, y could be considered to provide the necessary memory for the second delay.

One aesthetically pleasing feature of the representation (Fig. 7) is the symmetry. If a complex input, $u + jv$ were to be filtered, then v would be found to be applied to the lower summer.

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